Large-Amplitude Oscillations of Oblique Panels with an Initial Curvature

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Von Kármán field equations for flexible oblique plates with an initial curvature are extended to a dynamical case Using series representation of initial and additional deflections and Galerkin's procedure, the governing equation for an admissible mode time function is established Using this single assumed modal deflection, and assuming built-in edge free to move in the inplane directions, the following particular cases are discussed: buckling of an oblique plate under uniaxial compressive load, free linear vibrations of a square plate, large deflections of a uniformly loaded square plate, snap-through phenomena of a curved oblique plate under uniform transverse load, and free nonlinear vibrations A numerical example concerning a rhombic plate is discussed in more detail The well-known fact of a decrease of the period of nonlinear vibrations with an increasing amplitude is corroborated, this relation being less pronounced for larger sweep angles

Introduction

OBLIQUE panels find frequent application in technology of modern flying vehicles, mainly as standard structural elements of swept-back wings of airplanes They appear in the form of flexible plates shaped as skew parallelograms, initially slightly curved and supported at the periphery by stringers and ribs usually parallel to the direction of flight Because of their flexibility, buckling phenomena involving flat oblique panels have been investigated to certain extent in the last decade, mostly with regard to a uniform uniaxial inplane compression and shear (compare, e g , Refs 1–6) On the other hand, dynamical phenomena have attracted less attention, and in particular those which lead to oscillations with a large amplitude In fact, the author could not trace any paper pertinent to the latter case † Analysis of such problems, however, may be of importance, for instance, in connection with oscillations of swept-back wings at supersonic flight

As far as nonlinear oscillations of rectangular panels are concerned, Grigoliuk[7] in 1955 analyzed this problem for a specific type of support applying Galerkin's method to von Kármán's equations extended to the dynamical case In 1956, Chu and Herrmann[8] studied vibrations of freely supported plates by means of a perturbation procedure and the principle of conservation of energy Making use of an approximate analysis offered by the Berger method, Nash and Modeer,[9] in 1959, investigated the nonlinear behavior of vibrating rectangular plates, and in 1962 Hassert and Nowinski[10] discussed this problem with regard to an orthogonally anisotropic material and a simple support

In the present paper, an investigation is made of the transverse large-amplitude vibrations of oblique plates with double initial curvature Von Kármán field equations are extended to embrace the dynamical case, and Galerkin's procedure is used furnishing the governing equation for a lowest-mode time function Assuming a particular type of support, buckling of oblique plates under uniaxial compressive load and large transverse deflections under a uniform load are discussed Free linear and free nonlinear vibrations are investigated in more detail A numerical example involving a rhombic plate is given

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† It is likely that the search was not extensive enough

1 Formulation of the Problem

Let us consider an oblique initially doubly curved elastic plate of length a and width b, the angle of the sweptback being φ and the constant thickness of the plate being h (Fig 1) At the center of the plate let us locate the origin of two Cartesian coordinate frames: one rectangular x,y, and one oblique ξ,η related to each other by means of the mapping

$$\xi = x/\cos\varphi \qquad \eta = -tg\varphi x + y \tag{1 1}$$

Since, in the oblique coordinate system ξ,η, the Laplace operator has the form

$$\nabla_{\xi\eta}{}^2 = \frac{1}{\cos^2\varphi}\left[\frac{\partial^2}{\partial\xi^2} - 2\sin\varphi\,\frac{\partial^2}{\partial\xi\partial\eta} + \frac{\partial^2}{\partial\eta^2}\right] \tag{1 2}$$

the well-known von Kármán field equations (Ref 17, pp 348–351) extended to the dynamical case and to small initial curvature appear as

$$F(w,\phi) \equiv (D/h)\nabla^2\nabla^2 w - L(w,\phi) - L(w_0,\phi) - (q/h) + \rho(\partial^2 w/\partial t^2) = 0 \tag{1 3}$$

$$(1/E)\nabla^2\nabla^2\phi = -\tfrac{1}{2}L(w,w) - L(w_0,w) \tag{1 4}$$

where, for conciseness, the subscripts at the nabla operators have been omitted In the foregoing equations, D, as usual, stands for the bending rigidity of the plate, ρ for its mass density, $q = q(\xi,\eta)$ is the intensity of the external load, $w_0 = w_0(\xi,\eta)$ the initial deflection of the plate, $w = w(\xi,\eta,t)$ an additional deflection produced by the load, and

$$L(w,\phi) = \frac{1}{\cos^2\varphi}\left(\frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2\phi}{\partial\eta^2} + \frac{\partial^2 w}{\partial\eta^2}\frac{\partial^2\phi}{\partial\xi^2} - 2\,\frac{\partial^2 w}{\partial\xi\partial\eta}\frac{\partial^2\phi}{\partial\xi\partial\eta}\right) \tag{1 5}$$

We note that the skew membrane stresses σ_ξ, σ_η, and $\tau_{\xi\eta}$ may be expressed by means of the stress function $\phi(\xi,\eta)$ as

$$\sigma_\xi = (1/\cos\varphi)(\partial^2\phi/\partial\eta^2)$$
$$\sigma_\eta = (1/\cos\varphi)(\partial^2\phi/\partial\xi^2) \tag{1 6}$$
$$\tau_{\xi\eta} = -(1/\cos\varphi)(\partial^2\phi/\partial\xi\partial\eta)$$

In this form, the stresses (1 6) satisfy the equations of balance of forces in the directions tangent to the middle surfaces of the panel, provided the inertia terms inherent with the motion in the middle surface are disregarded Such an assumption has been corroborated (see, e g , Ref 8) and is commonly accepted in problems in which vibrations take

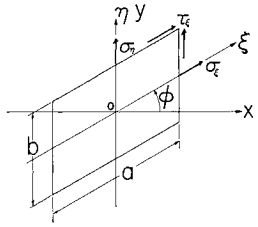

Fig 1 Geometry of the plate

place, principally in the direction perpendicular to the middle surface

To the governing field equations [(1 3) and (1 4)] we must adjoin the initial and boundary conditions to be discussed later This completes the formulation of the problem that is now reduced to the integration of a system of two nonlinear differential equations in combination with some prescribed conditions Apparently, the difficult problem under investigation may be solved only by using an approximate technique For this purpose, assume first, for definiteness, that the plate is built-in at the periphery, that is,

$$w[\pm(a/2),\eta,t] = 0 \qquad w[\xi,\pm(b/2),t] = 0$$
$$(\partial w/\partial\xi)[\pm(a/2),\eta,t] = 0 \qquad (\partial w/\partial\eta)[\xi,\pm(b/2),t] = 0 \tag{1 7}$$

at each time $t \geqslant 0$ We now represent the deflection of the plate in the following separable form:

$$w(\xi,\eta,t) = \sum_{\nu\,\mu} A_{\nu\mu}\tau_{\nu\mu}(t)\,\varphi_{\nu\mu}(\xi,\eta) \tag{1 8}$$

where $\tau_{\nu\mu}(t)$ are unknown functions of time only to be determined in the sequel, and $\varphi_{\nu\mu}(\xi,\eta)$ are suitably chosen spacial functions which satisfy the boundary conditions (1 7) ‡ In what follows, we pose, without relevant loss in generality,

$$\varphi_{\nu\mu}(\xi,\eta) = [1 + (-1)^{\nu+1}\cos\alpha_\nu\xi][1 + (-1)^{\mu+1}\cos\beta_\mu\eta] \tag{1 9}$$

where $\alpha_\nu = 2\pi\nu/a$ and $\beta_\mu = 2\pi\mu/b$

The initial deflection of the plate is expanded into the Fourier series

$$w_0(\xi,\eta) = \sum_{\nu\,\mu} B_{\nu\mu}\varphi^0{}_{\nu\mu}(\xi,\eta) \tag{1 10}$$

in which $B_{\nu\mu}$ clearly denote known constant coefficients, and the functions $\varphi^0{}_{\nu\mu}(\xi,\eta)$ are selected in the form analogous to (1 9) To avoid complications in writing, in what follows we confine ourselves to a single term in each series (1 8) and (1 10), thus obtaining

$$w(\xi,\eta,t) = A\tau(t)(1 + \cos\alpha\xi)(1 + \cos\beta\eta) \tag{1 11}$$

$$w_0(\xi,\eta) = A_0(1 + \cos\alpha\xi)(1 + \cos\beta\eta) \tag{1 12}$$

where now $\alpha = 2\pi/a$ and $\beta = 2\pi/b$ Of course, at a considerable expense of computational work, it is possible to employ more terms of the series (1 8) and (1 10) than one and to increase the generality as well as, to some extent, the accuracy of the results Nonetheless, even drastically truncated equations (1 11) and (1 12) may furnish some useful results of a specific nature although subject to limitations pertinent to a single mode approximation

2 Determination of Membrane Stresses

We now proceed to determine the stress function $\phi(\xi,\eta,t)$ and the membrane stresses in terms of ϕ For this purpose, we substitute Eqs (1 11) and (1 12) into the right-hand member of the governing equation (1 4) and find, with a little manipulation, the following particular integral of (1 4) suitable for the problem under investigation:

$$\phi(\xi,\eta,t) = -\tau^*\cos^2\varphi\left[\frac{\cos\alpha\xi}{\alpha^4} + \frac{\cos\beta\eta}{\beta^4} + \frac{\cos2\alpha\xi}{16\alpha^4} + \frac{\cos2\beta\eta}{16\beta^4} + \lambda_1(\cos\alpha\xi\cos\beta\eta - \varkappa_1\sin\alpha\xi\sin\beta\eta) + \lambda_2(\cos2\alpha\xi\cos\beta\eta - \varkappa_2\sin2\alpha\xi\sin\beta\eta) + \lambda_3(\cos\alpha\xi\cos2\beta\eta - \varkappa_3\sin\alpha\xi\times\sin2\beta\eta)\right] + \cos\varphi\left(p_\xi\frac{\eta^2}{2} + p_\eta\frac{\xi^2}{2} - p_{\xi\eta}\xi\eta\right) \tag{2 1}$$

In the preceding equation, p_ξ, p_η, and $p_{\xi\eta}$ denote oblique normal and shear stresses, constant throughout the region of the plate, eventually functions of time Moreover,

$$\tau^* = EA\alpha^2\beta^2[(A/2)\tau^2 + A_0\tau] \tag{2 2}$$

$$\lambda_1 = \frac{2}{(\alpha^2+\beta^2)^2 + 4\alpha^2\beta^2\sin^2\varphi - 4\alpha\beta(\alpha^2+\beta^2)\varkappa_1\sin\varphi}$$
$$\lambda_2 = \frac{1}{(4\alpha^2+\beta^2)^2 + 16\alpha^2\beta^2\sin^2\varphi - 8\alpha\beta(4\alpha^2+\beta^2)\varkappa_2\sin\varphi} \tag{2 3 1}$$
$$\lambda_3 = \frac{1}{(\alpha^2+4\beta^2)^2 + 16\alpha\beta^2\sin^2\varphi - 8\alpha\beta(\alpha^2+4\beta^2)\varkappa_3\sin\varphi}$$

$$\left.\begin{aligned}\varkappa_1 &= \frac{4\alpha\beta(\alpha^2+\beta^2)\sin\varphi}{(\alpha^2+\beta^2)^2 + 4\alpha\beta^2\sin^2\varphi}\\ \varkappa_2 &= \frac{8\alpha\beta(4\alpha^2+\beta^2)\sin\varphi}{(4\alpha^2+\beta^2)^2 + 16\alpha^2\beta^2\sin^2\varphi}\\ \varkappa_3 &= \frac{8\alpha\beta(\alpha^2+4\beta^2)\sin\varphi}{(\alpha^2+4\beta^2)^2 + 16\alpha^2\beta^2\sin^2\varphi}\end{aligned}\right\} \tag{2 3 2}$$

With these in mind, a substitution of (2 1) into (1 6) yields the components of the membrane stresses in the following final form:

$$\left.\begin{aligned}\sigma_\xi &= \tau^*\cos\varphi\left[\frac{\cos\beta\eta}{\beta^2} + \frac{\cos2\beta\eta}{4\beta^2} + \lambda_1\beta^2\cos\alpha\xi\cos\beta\eta - \varkappa_1\lambda_1\beta^2\sin\alpha\xi\sin\beta\eta + \lambda_2\beta^2\cos2\alpha\xi\cos2\beta\eta - \lambda_2\varkappa_2\beta^2\sin2\alpha\xi\sin\beta\eta + 4\lambda_3\beta^2\cos\alpha\xi\cos2\beta\eta - 4\lambda_3\varkappa_3\beta^2\sin\alpha\xi\sin2\beta\eta\right] + p_\xi\\ \sigma_\eta &= \tau^*\cos\varphi\left[\frac{\cos\alpha\xi}{\alpha^2} + \frac{\cos2\alpha\xi}{4\alpha^2} + \lambda_1\alpha^2\cos\alpha\xi\cos\beta\eta - \varkappa_1\lambda_1\alpha^2\sin\alpha\xi\sin\beta\eta + 4\lambda_2\alpha^2\cos2\alpha\xi\cos\beta\eta - 4\lambda_2\varkappa_2\alpha^2\sin2\alpha\xi\sin\beta\eta + \lambda_3\alpha^2\cos\alpha\xi\cos2\beta\eta - \lambda_3\varkappa_3\alpha^2\sin\alpha\xi\sin2\beta\eta\right] + p_\eta\\ \tau_{\xi\eta} &= \tau^*\alpha\beta\cos\varphi[\lambda_1\sin\alpha\xi\sin\beta\eta - \lambda_1\varkappa_1\cos\alpha\xi\cos\beta\eta + 2\lambda_2\sin2\alpha\xi\sin\beta\eta - 2\lambda_2\varkappa_2\cos2\alpha\xi\sin\beta\eta + 2\lambda_3\sin\alpha\xi\sin2\beta\eta - 2\lambda_3\varkappa_3\cos\alpha\xi\cos2\beta\eta] + p_{\xi\eta}\end{aligned}\right\} \tag{2 4}$$

It may be easily shown upon integration of Eqs (2 4) along the edges of the plate that the contribution of the first member at the right-hand side of Eqs (2 4), enclosed in brackets

‡ For future calculations, it is expedient to separate in (1 8) the constant coefficients $A_{\nu\mu}$ from the associated time functions

to the average stresses at the corresponding edge of the plate is nil Hence the constant stresses p_ξ, p_η, and $p_{\xi\eta}$ may be regarded as the average edge membrane stresses

To be more definite, assume that the clamping of the edges of the plate does not prevent free inplane motions This yields the following complementary boundary conditions:

$$p_\xi = p_\eta = p_{\xi\eta} = 0 \tag{2 5}$$

assumed in pertinent portions of the following development

3 Governing Time Equation

We now turn our attention to the only remaining field equation of motion (1 3) to which we apply the procedure of Galerkin (compare, e g , Ref 7 and Ref 11, p 58) We proceed in the following way Upon substituting from (1 11, 1 12, and 2 1) into (1 3), we multiply $F(w,\phi)$ by the spacial part of the coordinate function (1 11), that is, by $(1 + \cos\alpha\xi)\ (1 + \cos\beta\eta)$, and integrate the result over the region $-(a/2) \leqslant \xi \leqslant (a/2)$, $-(b/2) \leqslant \eta \leqslant (b/2)$ A longer but elementary calculation yields the following nonlinear differential equation of the second order for the time function τ:

$$9\rho A\frac{d^2\tau}{dt^2} + \left\{\frac{AD}{h\cos^4\varphi}\left[3\alpha^4 + 3\beta^4 + 2\alpha^2\beta^2(1 + 2\sin^2\varphi)\right] + \frac{3A}{\cos\varphi}\left(\alpha^2 p_\xi + \beta^2 p_\eta\right) + EA_0{}^2\alpha^2\beta^2\Omega\right\}\tau + \frac{3}{2}EA^2A_0\alpha^2\beta^2\Omega\tau^2 + \frac{1}{2}EA^3\alpha^2\beta^2\Omega\tau^3 - \frac{4q}{h} + \frac{3}{\cos\varphi}\left(\alpha^2 p_\xi + \beta^2 p_\eta\right)A_0 = 0 \tag{3 1 1}$$

in which

$$\Omega = \alpha^2\left[\frac{17}{8\beta^2} + (2\lambda_1 + \lambda_2 + 2\lambda_3)\beta^2\right] + \beta^2\left[\frac{17}{8\alpha^2} + (2\lambda_1 + 2\lambda_2 + \lambda_3)\alpha^2\right] + 2(\lambda_1 + \lambda_2 + \lambda_3)\alpha^2\beta^2 \tag{3 1 2}$$

providing the loading function $q(\xi,\eta,t)$ is independent of the space coordinates, thus representing a uniform loading eventually varying with time

At this stage, it is expedient to introduce dimensionless variables by means of the following definitions:

$$\left.\begin{aligned} &\sigma = h\alpha^2\left(\frac{E}{\rho}\right)^{1/2}t \qquad A^*\tau(t) = U(\sigma) \qquad A^* = 4\frac{A}{h} \\ &A_0{}^* = 4\frac{A_0}{h} \qquad \mu = \frac{\beta}{\alpha} = \frac{a}{b} \qquad p_\xi{}^* = \frac{p_\xi}{Eh^2\alpha^2} \\ &p_\eta{}^* = \frac{p\eta}{Eh^2\alpha^2} \qquad q^* = \frac{qa^4}{Eh^4} \end{aligned}\right\} \tag{3 2}$$

An inspection of (3 2) reveals that $A_0{}^*$ and A^* denote nondimensional initial and additional deflection at the center of the plate, respectively ($A_0{}^* = [w_0(0)/h]$, $A^* = [w(0)/h]$) Upon substitution from (3 2), the governing time equation (3 1) takes the following final form:

$$\frac{d^2U}{d\sigma^2} + u_1U + u_2U^2 + u_3U^3 + \frac{1}{3\cos\varphi}\left(p_\xi{}^* + \mu^2 p_\eta{}^*\right)A_0{}^* - \frac{q^*}{9\pi^4} = 0 \tag{3 3}$$

where

$$\left.\begin{aligned} u_1 &= \frac{1}{36}\left\{\frac{1}{3(1-\nu^2)\cos^4\varphi}\times\left[3 + 3\mu^4 + 2\mu^2(1 + 2\sin^2\varphi)\right] + \frac{1}{4}\mu^2\Omega(A_0{}^*)^2 + \frac{12}{\cos\varphi}\left(p_\xi{}^* + \mu^2 p_\eta{}^*\right)\right\} \\ u_2 &= \frac{1}{96}\mu^2\Omega A_0{}^* \qquad u_3 = \frac{1}{288}\mu^2\Omega \\ \Omega &= \mu^2\left(\frac{17}{8\mu^4} + 2\lambda_1{}^* + \lambda_2{}^* + \lambda_3{}^* + \frac{17}{8}\right) \end{aligned}\right\} \tag{3 4}$$

$$\lambda_1{}^* = \frac{2}{(1+\mu^2)^2 + 4\mu^2\sin^2\varphi - 4\mu\kappa_1(1+\mu^2)\sin\varphi}$$

$$\lambda_2{}^* = \frac{1}{(4+\mu^2)^2 + 16\mu^2\sin^2\varphi - 8\mu\kappa_2(4+\mu^2)\sin\varphi}$$

$$\lambda_3{}^* = \frac{1}{(1+4\mu^2)^2 + 16\mu^2\sin^2\varphi - 8\mu\kappa_3(1+4\mu^2)\sin\varphi}$$

Equation (3 3) provides a background for the entire discussion that follows For a more detailed development of the ideas, let us now specialize Eq (3 3) to refer to the following particular cases

A Buckling of a Flat Rectangular Plate with Built-in Edges under Inplane Compressive Load

In this case we set $A_0 = \varphi = q = 0$, $\tau = 1$, and, say, $p_\eta = 0$ Since a disclosure of the critical value of the compressive force $N_\xi = \sigma_\xi h$ represents a linear problem, we disregard the powers of the amplitude A^* higher than the first This yields, after returning to the dimensional form,

$$N_{\xi\,\mathrm{crit}} = -\frac{4\pi^2}{3}Da^2\left(\frac{3}{a^4} + \frac{3}{b^4} + \frac{2}{a^2b^2}\right) \tag{3 5}$$

in accord with the well-established result (compare, for instance, Timoshenko's, Ref 12, p 321)

For an oblique plate, the critical load becomes

$$N_{\xi\,\mathrm{crit}} = -\frac{4\pi^2}{3a^2\cos^3\varphi}\left[3(1+\mu^4) + 2\mu^2(1 + 2\sin^2\varphi)\right] \tag{3 6}$$

Apparently, the first approximation of the buckled middle surface of the plate furnished by Eq (1 11), as a by-product of the solution of the main problem, cannot be considered an adequate representation of the true, rather complex, shape of the middle surface in the oblique case Hence, the discrepancies between the result (3 6) and the results of more refined solutions such as derived, for instance, by Wittrick,[2] may be substantial A similar situation, however, has to be noted with regard to first approximations furnished by more rigorous techniques specifically designed for solution of the buckling problem §

§ For the values of the angle φ, equal to 15 and 30°, the error involved in (3 6) amounts to 15 5 and 57%, respectively, as compared with Ref 2 (data taken from the graph in Fig 2, reproduced in Ref 5) In Ref 6, for $\varphi = 35°$, the error of the first approximation exceeds 27% (this paper treats a free support of the plate) The sensitivity of the critical compressive stress (3 6) on the angle of sweep-back, in particular for larger values of φ, becomes apparent if one notices that as $\varphi \to \pi/2$, $N_{\xi\,\mathrm{crit}} \to \infty$, thus furnishing a steep rising curve

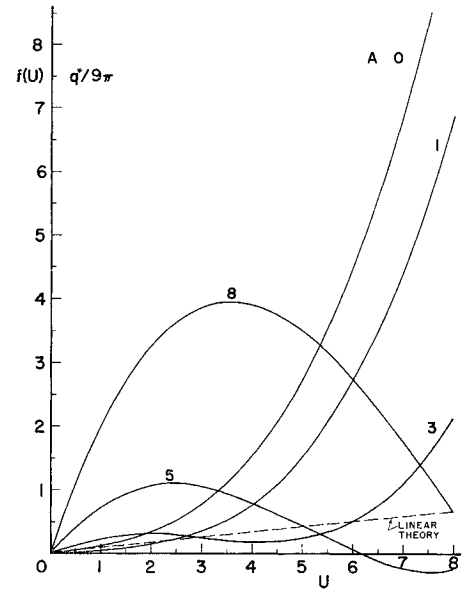

Fig 2 Load or control function vs deflection U for a square panel and various values of initial deflection A_0^*

B Free Linear Vibrations of a Flat Square Plate with Built-in Edges

In this case, we set, as in the previous case, $A_0 = q = 0$ and $\varphi = 0$ Furthermore, we disregard τ^2 and τ^3 and drop, of course, p_ξ and p_η Since $\alpha = \beta = 2\pi/a$, say, Eq (3 1) becomes simply

$$(d^2\tau/dt^2) + (128/9)(\pi^4 D/\rho h a^4)\tau = 0 \qquad (3\ 7)$$

From (3 7), the circular frequency of linear vibrations follows immediately as

$$\omega = (37\ 2/a^2)(D/h\rho)^{1/2} \qquad (3\ 8)$$

in accordance with the well-known result based upon the same assumed deflection form in Ref 13, p 375

C Large Deflections of a Uniformly Loaded Flat Rectangular Plate with Built-in Edges

In this particular case, we have to pose $A_0 = \varphi = 0$ and $\tau = 1$ Equation (3 3) presently becomes

$$\frac{U}{36}\left\{\frac{1}{3(1-\eta^2)}\,[3(1+\mu^2)+2\mu^2] + 12(p\xi^* + \mu^2 p\eta^*)\right\} + \frac{1}{288}\,\mu^2\Omega U^3 - \frac{q^*}{9\pi^4} = 0 \qquad (3\ 9)$$

in accordance with the result obtained, for instance, by Volmir [Ref 11, p 112, Eq (2 146)] ¶

4 Snap-Through Phenomenon under a Uniform Transverse Load

We now turn our attention to the general equation (3 3) in which the time derivative is disregarded and the symbol τ substituted for by 1 Suppose that an oblique plate with a small initial curvature is acted upon by a uniformly distributed load in the direction opposite to that of the initial deflection and that the edges of the plate are free, i e , $p_\xi^* = p_\eta^* = 0$ From (3 3) it is seen that

$$(q^*/9\pi^4) = u_1 U - u_2 U^2 + u_3 U^3 \qquad (4\ 1)$$

where A_0^* is considered to be positive Apparently, the sign preceding the term u_2U^2 in Eq (4 1) has been reversed in order to account for the fact that the direction of the load is opposite to that of the initial deflection, as assumed previously Relation (4 1) between the nondimensional intensity of load and the center deflection of the plate can be displayed graphically in a convectional way (compare, e g , Ref 11, p 94 and p 208) for various values of the amplitude of the initial deflection A_0^* as well as of the angle of sweepback φ Figure 2 illustrates this interdependence for a square panel It is known that, in a certain range of the load and beyond a certain value of the initial deflection, to each value of q^* correspond, in general, three different states of equilibrium of the plate

Denote the right-hand number of (4 1) by $f(U)$ Thus

$$f(U) = u_1 U - u_2 U^2 + u_3 U^3 \qquad (4\ 2)$$

and the extremal values of the load q^* occur for values of $U = U_i^{\,ext}$, $(i = 1,2)$, which satisfy the condition $f'(U) = 0$ After making use of relations (3 4) we get

$$U_{1\,2}^{\,ext} = A_0^*\left[1 \mp \left\{\frac{1}{3}\left[1 - \frac{8g(\varphi)}{\mu^2\Omega(A_0^*)^2}\right]\right\}^{1/2}\right] \qquad (4\ 3)$$

where

$$g(\varphi) = \frac{1}{3(1-\nu^2)\cos^2\varphi}\,[3 + 3\mu^4 + 2\mu^2(1 + 2\sin^2\varphi)] \qquad (4\ 3\ 1)$$

The equation $f'(U) = 0$ represents a quadratic parabola Q with a vertical axis, the convex side being turned toward the direction of negative ordinates, and the vertex located at

$$U_{\text{infl}} = A_0^* \qquad (4\ 4)$$

where $f''(U) = 0$ It follows that each curve $q^*/9\pi^4 - f(U) = 0$ has an inflection point at U equal to U_{infl} As long as the second term in the braces in (4 3) is large, that is, as long as the initial deflection is small and/or the angle of sweep-back is large, the parabola Q is soaring over the U axis and the $q^* = q^*(U)$ curves increase monotonically with U For $A_0^* = [8g(\varphi)/\mu^2\Omega]^{1/2}$ there is $U_1^{\,ext} = U_2^{ext}$, and this double root corresponds to a stationary value of $q^*(U)$ at which the curve $q^* = q^*(U)$ has an inflection point with a horizontal tangent The latter curve represents a greater lower bound for the curves $q^* = q^*(U)$, which display a loop associated with a multiple-valuedness of the inverse function $U = U(q^*)$ From then on, the parabolas Q intersect the U axis This is clearly visible, e g , in Fig 3 in which $A_0^* = 3$ corresponds to a value of A_0^* less than the critical value (associated with a horizontal tangent at the inflection point) while $A_0^* = 5$ is greater than the critical value of A_0^* Figures 2, 3, and 4 display the dependence of the reduced load

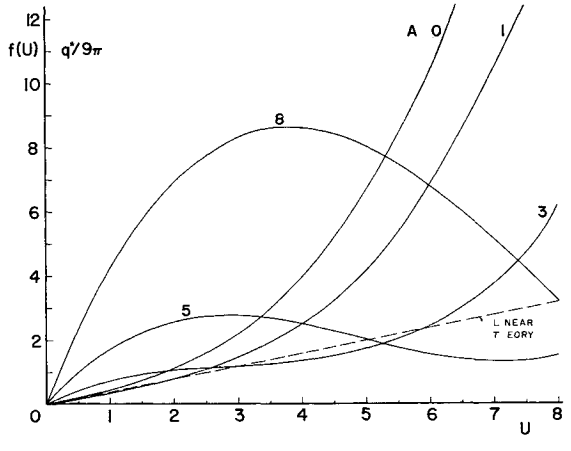

Fig 3 Load or control function vs deflection U for a rhombic panel ($\varphi = 45^\circ$) and various values of initial deflection A_0^*

¶ It should be noted that some misprints in this equation as well as in Sec 21 of Ref 11 need correction

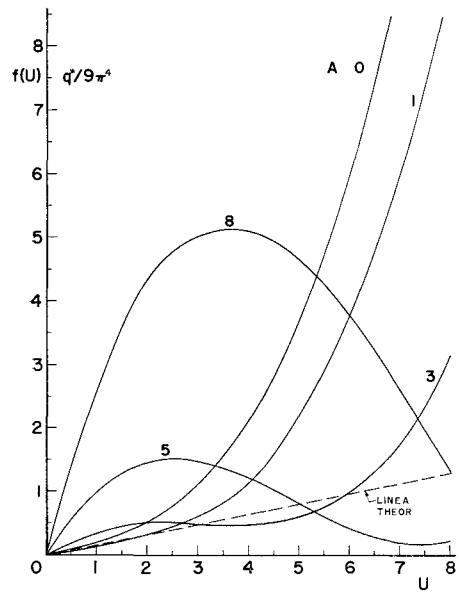

Fig 4 Load or control function vs deflection U for a rhombic panel ($\varphi = 30$) and various values of initial deflection A_0*

on the deflection parameter U for a rhombic panel and for various values of the dimensionless initial deflection A_0* as well as of the angle of sweep-back φ ** An inspection reveals the fact that, for a given initial deflection, the greater the value of φ, the greater the load which is required to produce a snap-through deflection However, the snap-through occurs at a similar value of additional deflection Moreover, the greater the angle φ, the greater the value of A_0* at which higher states of equilibrium may occur

5 Free Nonlinear Vibrations

We now proceed to discuss the general case of free nonlinear vibrations described by the equation

$$\ddot{U} + f(U) = 0 \tag{5 1}$$

with the restoring force (control term) $f(U)$ in the form of a cubic polynomial Although an explicit integration of (5 1) through the use of elliptic integrals and functions is still possible, some important information of a qualitative character may be obtained using geometrical discussion in the phase plane ††

We pose $V = \dot{U}$, and integrate (5 1) This yields the energy equation

$$(V^2/2) + F(U) = h \tag{5 2}$$

where

$$F(U) = (u_1/2)U^2 - (u_2/3)U^3 + (u_3/4)U^4 \tag{5 3}$$

is the potential energy of the system, and constant

$$h = (V_0^2/2) + F(U_0) \tag{5 4}$$

represents the total energy Note that $F(U) = \int f(U)dU$ and that, at some initial time $\sigma = \sigma_0$, there is $U = U_0$ and $V = V_0$ It is also worth recalling that the graphs of the first derivative of the potential energy function are displayed in Figs 2–4

As is known, in order to analyze the integral curves (of constant energy) it is necessary to determine the singular points of the differential equation (5 1) The singularities correspond to rest positions of equilibrium since they are characterized by the conditions

$$V = 0 \qquad f(U) = 0 \tag{5 5}$$

as displayed in a qualitative way in Fig 5 In terms of the potential energy $F(U)$, they occur for values of U satisfying the condition $F'(U) = 0$, that is, at the points of extrema of potential energy (cf Ref 14, p 49), and the sign of $f'(U)$ establishes the type of equilibrium For future use, we note that, in view of (2 3), all the coefficients λ_i* ($i = 1,2,3$) are also positive Using this fact, we find that

$$f''(U) = (\mu^2\Omega/48)(U - A_0^*) \tag{5 6}$$

is negative in every sufficiently small vicinity of the origin, provided $A_0^* \neq 0$ Hence, the plate reveals a soft spring behavior near the origin for all values of the initial curvature except that equal to zero, when the spring behavior becomes hard We write now (5 1) in the form

$$dV/dU = [-f(U)/V] \tag{5 7}$$

and compare (5 7) with the standard form

$$\frac{dV}{dU} = \frac{aU + bV + G_2(U,V)}{cU + dV + H_2(U,V)} \tag{5 8}$$

for application of the criteria of Poincaré (G_2 and H_2 vanishing like $U^2 + V^2$ as $U,V \to 0$) In the present case, we have

$$a = -u_1 \qquad b = c = 0 \qquad d = 1 \tag{5 9 1}$$

so that the discriminant

$$D \equiv (b - c)^2 + 4ad = -4u_1 < 0 \tag{5 9 2}$$

and

$$b + c = 0 \tag{5 9 3}$$

Bearing in mind that the singularities occur for values of $U = U_{i\,\mathrm{sing}}$ ($i = 1, 2, 3$), satisfying $F'(U) \equiv f(U) = 0$ (in other words, at points corresponding to points of intersection, or tangency, of the control characteristic and the U axis), we obtain easily

$$U_{1\,\mathrm{sing}} = 0$$
$$U_{2\,3\,\mathrm{sing}} = \frac{A_0^*}{2}\left\{3 \mp \left[1 - \frac{32g(\varphi)}{\mu^2\Omega(A_0^*)^2}\right]^{1/2}\right\} \tag{5 10}$$

where apparently

$$U_{1\,\mathrm{sing}} \leqslant U_{2\,\mathrm{sing}} \leqslant U_{3\,\mathrm{sing}} \tag{5 11}$$

An inspection of (5 10) reveals the fact that for large values of the second term in the brackets [Eq (5 10)], that is, for small initial curvatures and/or for large values of the angle of sweep-back, there exists only one singular point located

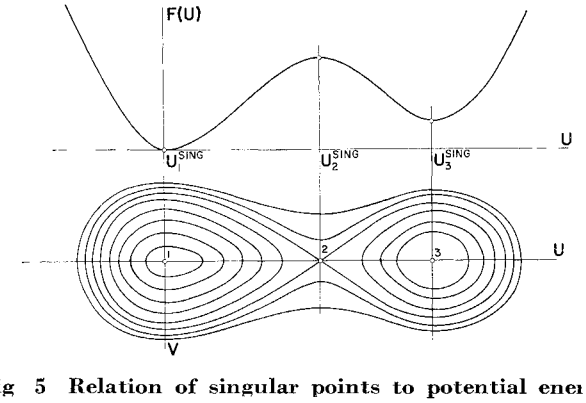

Fig 5 Relation of singular points to potential energy

** Computations were provided by R Beck student at the University of Delaware

†† Compare, e g , the books by Stoker (Ref 14, Chap 3) or McLachlan (Ref 15, Chap 9) An extensive analysis along these lines is given for nonlinear vibrations of rods and shallow axially symmetric shells in Ref 16 The following discussion is modeled on all this work

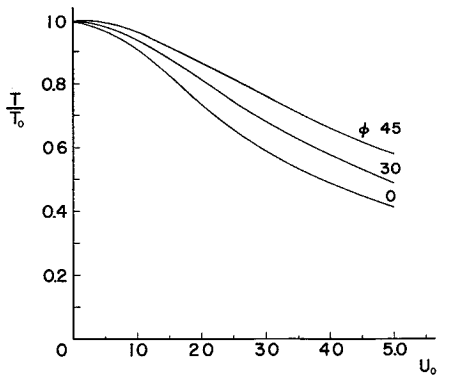

Fig 6 Relative period of nonlinear vibrations vs amplitude for a rhombic plate and various values of the angle φ

at the origin of coordinates For

$$A_0^* = 4/\mu(2g(\varphi)/\Omega)^{1/2} \tag{5 12}$$

there occurs a second singular point, besides $U = 0$, located at $U = \frac{3}{2}A_0^*$ Finally, if A_0^* is greater than the right-hand member of (5 12), we list three separate singular points

We now proceed to a classification of singularities of Eq (5 1)

A Singular Point $U = U_1^{sing}$, $V = 0$

Since, in this case, the discriminant is negative, the condition $b + c = 0$ no longer suffices to distinguish between a center and a spiral (see, e g , Ref 14, p 44) The derivative of the control term, however, df/dU, is positive at the origin so that the singular point is a center (compare point 1 in Fig 5; see also Ref 15, p 205)

B Singular Point $U = U_2^{ing}$, $V = 0$

We observe that the change of origin from $U = 0$ to U_2^{sing} obtained by setting $U = U_2^{ing} + z$ replaces (5 7) by

$$\frac{dV}{dz} = -\frac{u_3U_2^{ing}(U_2^{ing} - U_3^{sing})z + \dots}{V} \tag{5 13}$$

where the dots stand for higher-order terms in z which are disregarded But, in the present case,

$$D = 4a = 4u_3U_2^{ing}(U_3^{ing} - U_2^{ing}) > 0 \tag{5 14 1}$$

and

$$\Delta = ad - bc = a > 0 \tag{5 14 2}$$

Hence, the singularity is a saddle (compare point 2 in Fig 5) Since, specifically,

$$\Delta = U_2^{ing}(u_2^2 - 4u_1u_3)^{1/2} \tag{5 14 3}$$

then, if $u_2 = 2(u_1u_3)^{1/2}$, the saddle degenerates into a cusp

C Singular point $U = U_3^{sing}$, $V = 0$

The argument runs now closely parallel to that of the preceding case A change of origin replaces (5 7) by

$$\frac{dV}{dz} = -\frac{u_3U_3^{sing}(U_3^{ing} - U_2^{ing})z + \dots}{V} \tag{5 15}$$

with

$$D = 4a = 4u_3U_3^{sing}(U_2^{ing} - U_3^{ing}) < 0 \tag{5 16 1}$$

and

$$b + c = 0 \tag{5 16 2}$$

The singular point may be thus either a center or a spiral

It may be easily verified, however, that

$$\left[\frac{df}{dz}\right]_{=0} = U_3^{ing}(u_2^2 - 4u_1u_3)^{1/2} > 0 \tag{5 17}$$

so that, in general, the singularity is a center (compare point 3 in Fig 5) and for $u_2 = 2(u_1u_3)^{1/2}$ becomes a cusp In the latter case, the second and third singularities coincide, and the potential energy curve suffers, at the corresponding point, an inflection with a tangent directed horizontally

As is known, the period of oscillations T^* associated with either of the singularities of Eq (5 1) may be calculated by the line integral

$$T^* = \oint dU/V \tag{5 18}$$

to be taken along a closed energy curve in the direction of increasing time parameter t

For a flat plate, say, rhombic for definiteness, the period of free nonlinear vibrations may be easily determined by an explicit integration Recalling Eqs (3 3), and noting that in the present case $A_0^* = 0$ and $\mu = 1$, leads to

$$(d^2U/d\sigma^2) + u_1U + u_3U^3 = 0 \tag{5 19}$$

where now

$$u_1 = \frac{2 + \sin^2\varphi}{27(1 - \nu^2)\cos^4\varphi} \qquad u_3 = \frac{\lambda_1^* + \lambda_2^* + \frac{17}{8}}{144} \tag{5 20}$$

$$\lambda_1^* = \frac{1 + \sin^2\varphi}{2(1 - \sin^2\varphi)^2} \qquad \lambda_2^* = \frac{25 + 16\sin^2\varphi}{(25 - 16\sin^2\varphi)^2}$$

By virtue of (5 2), the integral curves on the phase plane become

$$V^2 + u_1U^2 + (u_3/2)U^4 = 2h \tag{5 21}$$

and they are symmetric with respect to both the coordinate axes The only real singular point is that located at the origin and is a center

Take as initial conditions, at $t = 0$, $A^* = U_0$ (so that $\tau(0) = 1$) and $V = \dot{U} = 0$ Hence U_0 represents the amplitude of oscillations It is easily shown that, in the case considered, the period of the nonlinear oscillations becomes

$$T^* = 4/(u_1 + u_3U_0^2)^{1/2}F[k,(\pi/2)] \tag{5 22}$$

where $F(k, \pi/2)$ is a complete elliptic integral of the first kind, with modulus $k^2 = \frac{1}{2}[1 + (u_1/u_3)U_0^{-2}]$ (compare Ref 15, p 27) Since the period of free linear oscillations is

$$T_0 = 2\pi/(u_1)^{1/2} \tag{5 23}$$

the relative period of nonlinear oscillations appears finally as

$$\frac{T^*}{T_0} = \frac{2}{\pi[1 + (u_3/u_1)U_0^2]^{1/2}}F(k,\pi/2) \tag{5 24}$$

An inspection of (5 24) reveals the known fact that in virtue of the multiplier $[1 + (u_3/u_1)U_0^2]^{-1/2}$, the periodic time of nonlinear vibrations decreases with an increase of the amplitude U_0 Figure 6 depicts the dependence of the relative period of free nonlinear vibrations on the amplitude of vibrations U_0 for various values of the angle of sweep-back φ It is worth noticing that the corresponding curves become flatter for higher values of φ, so that a sweep-back of a plate makes the phenomenon of decreasing period less pronounced However, even for the largest value of φ analyzed, that is, for $\varphi = 45°$, the decrease of the period of vibrations is rather sharp, and for the amplitude, $U_0 = 3$ reaches nearly 25% Clearly, a decrease of φ and an increase of U_0 influence the relative period T^*/T_0 in the opposite directions Thus, by a suitable combination of the values of φ and U_0, it is possible to set up the value of the period of vibrations on some desired

level Let us note again that the foiegoing results concern a single mode, coiresponding to what we consider to be approximately a fundamental mode [compare Eq (1 11)] Thus, until a more accurate than one term approximation is known, one has to bear in mind the limitations of the present solution

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